CHAPTER ONE

Introduction

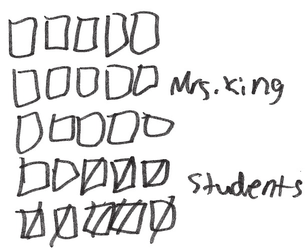
Why You Need to Teach Students to Mathematize

Imagine you are a new teacher. You are teaching fifth grade at a new school and are eager to get to know your students—their interests, skills, and how prepared they are to meet the challenges of fifth grade. You have just emerged from your teacher education program knowing various approaches you have seen modeled in classrooms and described in the literature, some of which you have tried with varying degrees of success. You aren't sure what approaches you want to use but are excited about challenging your students, introducing the rigor you have read so much about. But first, you need to know what your students can and can't do.

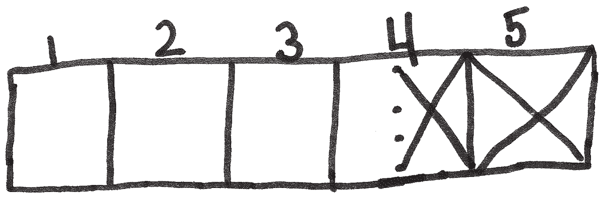
You decide to start with a couple of word problems, ones that involve relatively simple mathematical operations:

> *Mrs. King has 25 books to give to 8 students for summer reading after grade 4. If each student gets the same number of books, how many will she have left?*
>
> *Richard measured and packed enough flour to make brownies and a cake during the week at a cabin. He is baking the brownies first and the cake later. He has 5 cups of flour. The brownie recipe calls for $1\frac{1}{2}$ cups of flour, and whatever is left he will use for the cake. How much flour is left to use for the cake recipe?*

You circulate around the room, noting who draws pictures, who writes equations, and who uses the manipulatives you have put at the center of the table groups. While some students take their time, quite a few move quickly. Their hands go up eagerly, indicating they have solved the problems. As you check their work, one by one, you notice most of them got the first problem wrong, writing the equation $25 - 8 = 17$. Some even include a sentence saying, "Mrs. King will have 17 books left." Only one student in this group draws a picture. It looks like this:



Even though the second problem demands an understanding of fractions, a potentially complicating feature, most of these same students solve the second problem correctly. They write the equation $5-1\frac{1}{2}$ and are generally able to find the correct solution of $3\frac{1}{2}$ cups of flour left for the cake. You notice a few students use the fraction tiles available at tables to help them solve this problem. A number of students draw pictures for this problem. They often look something like this:



To learn more about how your students went wrong with the first problem, you call them to your desk one by one and ask about their thinking. A pattern emerges quickly. All of the students you talk to zeroed in on two key elements of the problem: (1) the total quantity of books the teacher started with and (2) how many were left. One student tells you, "*Left* always means to subtract. I learned that a long time ago." Clearly, she wasn't the only student who read the word *left* and assumed she had to subtract. This assumption, which led students astray in problem 1, luckily worked for these students in problem 2, where simple subtraction yielded a correct answer.

Problem-Solving Strategies Gone Wrong

In our work with teachers, we often see students being taught a list of "key words" that are linked to specific operations. Students are told, "Find the key word and you will know whether to add, subtract, multiply, or divide." Charts of key words often hang on classroom walls. Key words are a strategy that works often enough that teachers continue to rely on them. As we saw in the book-distribution problem, however, not only are key words not enough to solve a problem, but they also can easily lead students to an incorrect operation or to a single operation when multiple operations need to come into play. As the book problem reveals, different operations could successfully be called upon, depending on how the student approaches the problem—using division or even addition to distribute the books evenly, then determining how many remain. Subtraction could even be used, but it would not be the simple one-step subtraction operation we saw in the student's drawing, the one another student associated with the key word *left*.

Let's return to your imaginary classroom. Having seen firsthand the limitations of key words—a strategy you had considered using—where to begin? What approach to use? A new colleague has a suggestion. She agrees that relying on only key words can be too limiting. Instead, she is an enthusiastic proponent of a procedure called CUBES, which stands for teaching students these steps:

Circle the numbers

Underline important information

Box the question

Eliminate unnecessary information

Solve and check

She tells you that whenever she introduces a new kind of word problem, she walks students through the CUBES protocol using a "think-aloud," sharing how she is using the process to take apart the problem to find what to focus on. That evening, as you settle down to plan, you decide to walk through some problems like the book-distribution problem using CUBES. Circling the numbers is easy enough. You circle 25 (students), 8 (books), wondering briefly what students might do with the 4. Perhaps you will leave it out.

Then you tackle "important information." What is important here in this problem? Maybe the verb, that the teacher is *giving* students books. Certainly, it's important that all students get the same number of books. You box the question, but unfortunately the question contains that problematic word, *left*.

If you think this procedure has promise as a way to guide students through an initial reading of the problem, but leaves out how to help students develop a genuine understanding of the problem, you would be correct.

What is missing from procedural strategies such as CUBES and strategies such as key words, is—in a word—*mathematics* and the understanding of where it lives within the situation the problem is presenting. Rather than helping students learn and practice quick ways to enter a problem, we need to focus our instruction on helping them develop a deep understanding of the mathematical principles behind the operations and how they are expressed in the problem. They need to learn to *mathematize*.

What Is Mathematizing? Why Is It Important?

Mathematizing is the uniquely human process of constructing meaning in mathematics (from Freudenthal, as cited in Fosnot & Dolk, 2001). Meaning is constructed and expressed by a process of noticing, exploring, explaining, modeling, and convincing others of a mathematical argument. When we teach students to mathematize, we are essentially teaching them to take their initial focus off specific numbers and computations and put their focus squarely on the actions and relationships expressed in the problem, what we will refer to throughout this book as the **problem situation**. At the same time, we are helping students see how these various actions and relationships can be expressed and the different operations that can be used to express them. If students understand, for example, that equal-groups division problems, as in the book-distribution problem, involve fair sharing and that one way sharing can be expressed is by dividing, then they can learn where and how to apply the numbers in the problem, in order to develop an appropriate equation. If we look at problems this way, then finding a **solution** involves connecting the problem's context to its general kind of problem situation and to the operations that go with it. The rest is computation.

Making accurate and meaningful connections between different problem situations and the operations that can fully express them requires **operation sense**. Students with a strong operation sense

- Understand and use a wide variety of models of operations beyond the basic and

Mathematizing: The uniquely human act of modeling reality with the use of mathematical tools and representations.

Problem situation: The underlying mathematical action or relationship found in a variety of contexts. Often called "problem type" for short.

Solution: A description of the underlying problem situation along with an approach (or approaches) to finding an answer to the question.

Operation sense: Knowing and applying the full range of work for mathematical operations (for example, addition, subtraction, multiplication, and division).

Intuitive model of an operation: An intuitive model is "primitive," meaning that it is the earliest and strongest interpretation of what an operation, such as multiplication, can do. An intuitive model may not include all the ways that an operation can be used mathematically.

intuitive models of operations (Fischbein, Deri, Nello, & Marino, 1985)

- Use appropriate representations of actions or relationships strategically
- Apply their understanding of operations to any quantity, regardless of the class of number
- Can mathematize a situation, translating a contextual understanding into a variety of other mathematical representations

FOCUSING ON OPERATION SENSE

Many of us may assume that we have a strong operation sense. After all, the four operations are the backbone of the mathematics we were taught from day one in elementary school. We know how to add, subtract, multiply, and divide, don't we? Of course we do. But a closer look at current standards reveals nuances and relationships within these operations that many of us may not be aware of, may not fully understand, or may have internalized so well that we don't recognize we are applying an understanding of them every day when we ourselves mathematize problems both in real life and in the context of solving word problems. For example, current standards ask that students develop conceptual understanding and build procedural fluency in four kinds of addition/subtraction problems, including Add-To, Take-From, Compare, and what some call Put Together/Take Apart (we will refer to this category throughout the book as Part-Part-Whole). Multiplication and division have their own unique set of problem types as well. On the surface, the differences between such categories may not seem critical. But we argue that they are. Only by exploring these differences and the relationships they represent can students develop the solid operation sense that will allow them to understand and mathematize word problems and any other problems they are solving, whatever their grade level or the complexity of the problem. It does not mean that students should simply memorize the problem types. Instead they should have experience exploring all of the different problem types through word problems and other situations. Operation sense is not simply a means to an end. It has value in helping students naturally come to see the world through a mathematical lens.

USING MATHEMATICAL REPRESENTATIONS

Problem context: The specific setting for a word problem.

Mathematical representation: A depiction of a mathematical situation using one or more of these modes or tools: concrete objects, pictures, mathematical symbols, context, or language.

What would such instruction—instruction aimed at developing operation sense and learning how to mathematize word problems—look like? It would have a number of features. First it would require that we give students time to focus and explore by doing fewer problems, making the ones they do count. Next, it would facilitate students becoming familiar with various ways to represent actions and relationships presented in a **problem context**. We tend to think of solving word problems as beginning with words and moving toward number sentences and equations in a neat linear progression. But as most of us know, this isn't how problem solving works. It is an iterative and circular process, where students might try out different representations, including going back and rewording the problem, a process we call telling "the story" of the problem. The model that we offer in this book is based on this kind of active and expanded exploration using a full range of **mathematical representations**. Scholars who study mathematical modeling and problem solving identify five modes of representation: verbal, contextual, concrete, pictorial, and symbolic representations (Lesh, Post, & Behr, 1987).

VERBAL A problem may start with any mode of representation, but a word problem is first presented verbally, typically in written form. After that, verbal representations can serve many uses as students work to understand the actions and relationships in the problem situation. Some examples are restating the problem; thinking aloud; describing the math operations in words rather than symbols; and augmenting and explaining visual and physical representations including graphs, drawings, base 10 blocks, fraction bars, or other concrete items.

CONTEXTUAL The contextual representation is simply the real-life situation that the problem describes. Prepackaged word problems are based on real life, as is the earlier book-distribution problem, but alone they are not contextual. Asking students to create their own word problems based on real-life contexts will bring more meaning to the process and will reflect the purposes of mathematics in real life, such as when scientists, business analysts, and meteorologists mathematize contextual information in order to make predictions that benefit us all. This is a process called **mathematical modeling**, which Garfunkel and Montgomery (2019) define as the use of "mathematics to represent, analyze, make predictions or otherwise provide insight into real-world phenomena."

Mathematical modeling: A process that uses mathematics to represent, analyze, make predictions or otherwise provide insight into real-world phenomena.

CONCRETE Using physical representations such as blocks, concrete objects, and real-world items (for example, money, measuring tools, or items to be measured such as beans, sand, or water), or acting out the problem in various ways, is called **modeling**. Such models often offer the closest and truest representation of the actions and relationships in a problem situation.

Modeling: Creating a physical representation of a problem situation.

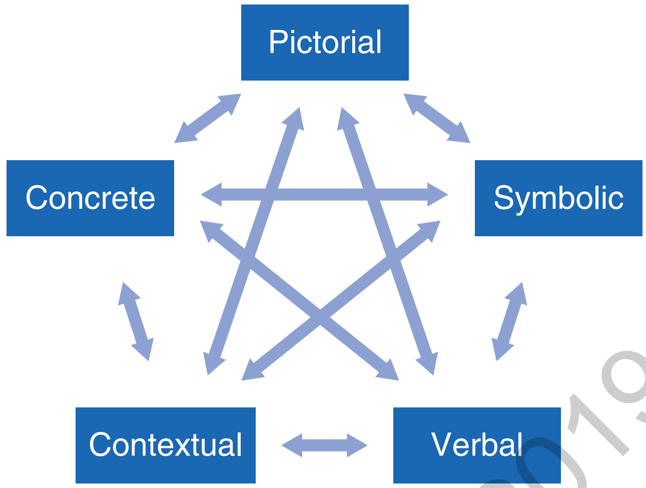
PICTORIAL Pictures and diagrams can illustrate and clarify the details of the actions and relationships in ways that words and even physical representations cannot. Using dots and sticks, bar models, arrows to show action, number lines, boxes to show regrouping, and various graphic organizers helps students see and conceptualize the nature of the actions and relationships.

SYMBOLIC Symbols can be operation signs ($+$, $-$, $\times$, $\div$), relational signs ($=$, $<$, $>$), variables (typically expressed as x, y, a, b, etc.), or a wide variety of symbols used in later mathematics (k, ∞, ϕ, π, etc.). Even though numerals are familiar, they are also symbols representing values (2, 0.9, $\frac{1}{2}$, 1,000).

There are two things to know about representations that may be surprising. First, mathematics can be shared *only* through representations. As a matter of fact, it is impossible to share a mathematical idea with someone else without sharing it through a representation! If you write an equation, you have produced a *symbolic* representation. If you describe the idea, you have shared a *verbal* representation. Representations are not solely the manipulatives, pictures, and drawings of a mathematical idea: They are any mode that communicates a mathematical idea between people.

Second, the strength and value of learning to manipulate representations to explore and solve problems is rooted in their relationship to one another. In other words, the more students can learn to move deftly from one representation to another, translating and/or combining them to fully illustrate their understanding of a problem, the deeper will be their understanding of the operations. Figure 1.1 reveals this interdependence. The five modes of representation are all equally important and deeply interconnected, and they work synergistically. In the chapters that follow, you will see how bringing multiple and synergistic representations to the task of problem solving deepens understanding.

FIGURE 1.1 FIVE REPRESENTATIONS: A TRANSLATION MODEL



Source: Adapted from Lesh, Post, and Behr (1987).

Teaching Students to Mathematize

As we discussed earlier, learning to mathematize word problems to arrive at solutions requires time devoted to exploration of different representations with a focus on developing and drawing on a deep understanding of the operations. We recognize that this isn't always easy to achieve in a busy classroom, hence, the appeal of the strategies mentioned at the beginning of the chapter. But what we know from our work with teachers and our review of the research is that, although there are no shortcuts, structuring exploration to focus on actions and relationships is both essential and possible. Doing so requires three things:

1. Teachers draw on their own deep understanding of the operations and their relationship to different word problem situations to plan instruction.
2. Teachers use a model of problem solving that allows for deep exploration.
3. Teachers use a variety of word problems throughout their units and lessons, to introduce a topic and to give examples during instruction, not just as the "challenge" students complete at the end of the chapter.

In this book we address all three.

BUILDING YOUR UNDERSTANDING OF THE OPERATIONS AND RELATED PROBLEM SITUATIONS

The chapters that follow explore the different operations and the various kinds of word problems—or problem situations—that arise within each. To be sure that all of the problems and situational contexts your students encounter are addressed, we drew on a number of sources, including the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010), the work done by the Cognitively Guided Instruction projects (Carpenter, Fennema, & Franke, 1996), earlier research, and our own work with teachers to create tables, one for addition and subtraction situations (Figure 1.2) and another for multiplication and division situations (Figure 1.3). Our versions of the problem situation tables represent the language we have found to resonate the most with teachers and students as they make sense of the various problem types, while still accommodating the most comprehensive list of categories. These tables also appear in the Appendix at the end of the book.

NOTES

FIGURE 1.2 ADDITION AND SUBTRACTION PROBLEM SITUATIONS

ACTIVE SITUATIONS

	Result Unknown	Change Addend Unknown	Start Addend Unknown	
Add-To	Paulo counted out 75 crayons and put them in the basket. Then he found 23 more crayons under the table. He added them to the basket. How many crayons are now in the basket? $75 + 23 = x$ $23 = x - 73$	Paulo counted out 75 crayons and put them in the basket. Then he found some more crayons under the table. He added them to the basket and now there are 98 crayons in the basket. How many crayons were under the table? $75 + x = 98$ $75 = 98 - x$	Paulo was organizing the crayons at his table. He found 23 crayons under the table and added them to the basket. When he counted, there are now 98 crayons in the basket. How many crayons were in the basket before Paulo looked under the table for crayons? $x + 23 = 98$ $98 - 23 = x$	
Take-From	There are 26 students in Mrs. Amadi's class. 15 left to get ready to play in the band at the assembly. How many students are not in the band? $26 - 15 = x$ $15 + x = 26$	There are 26 students in Mrs. Amadi's class. After the band students left the class for the assembly, there were 11 students still in the classroom. How many students are in the band? $26 - x = 11$ $x + 11 = 26$	15 band students left Mrs. Amadi's class to get ready to play in the assembly. There were 11 students left in the classroom. How many students are in Mrs. Amadi's class? $x - 15 = 11$ $15 + 11 = x$	

RELATIONSHIP (NON-ACTIVE) SITUATIONS

	Total Unknown	One Part Unknown		Both Parts Unknown
Part-Part-Whole	The 4th grade held a vote to decide where to go for the annual field trip. 32 students voted to go to the ice skating rink. 63 voted to go to the local park. How many students are in the 4th grade? $32 + 63 = x$ $x - 63 = 32$	The 4th grade held a vote to decide where the 95 students should go for their annual field trip. 32 students voted to go to the ice skating rink. The rest chose the local park. How many voted to go to the park? $32 + x = 95$ $x = 95 - 32$		The 4th grade held a vote to decide where the 95 students should go for their annual field trip. Some students voted to go to the ice skating rink and others voted to go to the local park. What are some possible combinations of votes? $x + y = 95$ $95 - x = y$

	Difference Unknown	Greater Quantity Unknown	Lesser Quantity Unknown	
Additive Comparison	Jessie and Roberto both collect baseball cards. Roberto has 53 cards and Jessie has 71 cards. How many fewer cards does Roberto have than Jessie? $53 + x = 71$ $53 = 71 - x$	Jessie and Roberto both collect baseball cards. Roberto has 53 cards and Jessie has 18 more cards than Roberto. How many baseball cards does Jessie have? $53 + 18 = x$ $x - 18 = 53$	Jessie and Roberto both collect baseball cards. Jessie has 71 cards and Roberto has 18 fewer cards than Jessie. How many baseball cards does Roberto have? $71 - 18 = x$ $x + 18 = 71$	

FIGURE 1.3 MULTIPLICATION AND DIVISION PROBLEM SITUATIONS

ASYMMETRICAL (NON-MATCHING) FACTORS				
	Product Unknown	**Multiplier (Number of Groups) Unknown**	**Measure (Group Size) Unknown**	
Equal Groups (Ratio/Rate)*	Mayim has 8 vases to decorate the tables at her party. She places 3 flowers in each vase. How many flowers does she need? $8 \times 3 = x$ $x \div 8 = 3$	Mayim has some vases to decorate the tables at her party. She places 3 flowers in each vase. If she uses 24 flowers, how many vases does she have? $x \times 3 = 24$ $x = 24 \div 3$	Mayim places 24 flowers in vases to decorate the tables at her party. If there are 8 vases, how many flowers will be in each vase? $8 \times x = 24$ $24 \div 8 = x$	
	Resulting Value Unknown	**Scale Factor (Times as many) Unknown**	**Original Value Unknown**	
Multiplicative Comparison	Amelia's dog is 5 times older than Wanda's 3 year-old dog. How old is Amelia's dog? $5 \times 3 = x$ $x \div 5 = 3$	Sydney has $15 to spend at the movies. Her sister has $5. How many times more money does Sydney have than her sister has? $x \times 5 = 15$ $5 = 15 \div x$	Mrs. Smith has 15 puzzles in her classroom. That is 3 times the number of puzzles in Mr. Jackson's room. How many puzzles are in Mr. Jackson's room? $3 \times x = 15$ $15 \div 3 = x$	
SYMMETRICAL (MATCHING) FACTORS				
	Product Unknown	**One Dimension Unknown**		**Both Dimensions Unknown**
Area/Array	Bradley bought a new rug for the hallway in his house. One side measured 5 feet and the other side measured 8 feet. How many square feet does the rug cover? $5 \times 8 = x$ $x \div 8 = 5$	The 40 members of the student council lined up on the stage to take yearbook pictures. The first row started with 8 students and the rest of the rows did the same. How many rows were there? $8 \times x = 40$ $x = 40 \div 8$		Daniella was building a house foundation using her building blocks. She started with 40 blocks. How many blocks long and wide could the foundation be? $x \times y = 40$ $40 \div x = y$
	Sample Space (Total Outcomes) Unknown	**One Factor Unknown**		**Both Factors Unknown**
Combinations** (Fundamental Counting Principle)	Karen has 3 shirts and 7 pairs of pants. How many unique outfits can she make? $3 \times 7 = x$ $3 = x \div 7$	Evelyn says that she can make 21 unique and different ice cream sundaes using just ice cream flavors and toppings. If she has 3 flavors of ice cream, how many kinds of toppings does Evelyn have? $3 \times x = 21$ $21 \div 3 = x$		Audrey can make 21 different fruit sodas using the machine at the restaurant. How many different flavorings and sodas could there be? $x \times y = 21$ $x = 21 \div y$

*Equal Groups problems, in many cases, are special cases of a category that includes all ratio and rate problem situations. Distinguishing between the two categories is often a matter of interpretation. The Ratio and Rates category, however, becomes a critically important piece of the middle school curriculum and beyond so the category is referenced here. It will be developed more extensively in the grades 6–8 volume of this series.
**Combinations are a category addressed in middle school mathematics standards. They are introduced briefly in chapter 8 for illustration purposes only and will be developed more extensively in the grades 6–8 volume of this series.

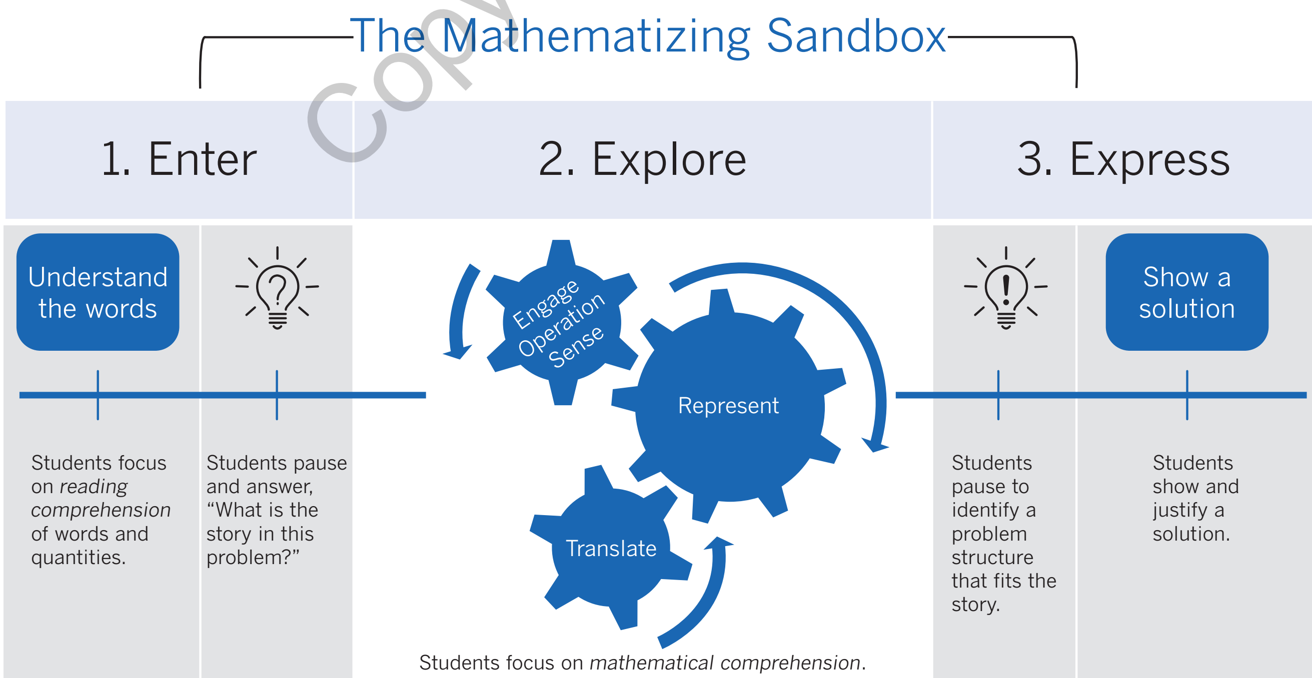
In the chapters—each of which corresponds to a particular problem situation and a row on one of the tables—we walk you through a problem-solving process that enhances your understanding of the operation and its relationship to the problem situation while modeling the kinds of questions and explorations that can be adapted to your instruction and used with your students. Our goal is *not* to have students memorize each of these problem types or learn specific procedures for each one. Rather, our goal is to help you enhance your understanding of the structures and make sure your students are exposed to and become familiar with them. This will support their efforts to solve word problems with understanding—through mathematizing.

In each chapter, you will have opportunities to stop and engage in your own problem solving in the workspace provided. We end each chapter with a summary of the key ideas for that problem situation and some additional practice that can also be translated to your instruction.

PLAYING IN THE MATHEMATIZING SANDBOX: A PROBLEM-SOLVING MODEL

To guide your instruction and even enhance your own capacities for problem solving, we have developed a model for solving word problems that puts the emphasis squarely on learning to mathematize (Figure 1.4). The centerpiece of this model is what we call the "mathematizing sandbox," and we call it this for a reason. The sandbox is where children explore and learn through play. Exploring, experiencing, and experimenting by using different representations is vital not only to developing a strong operation sense but also to building comfort with the problem-solving process. Sometimes it is messy and slow, and we as teachers need to make room for it. We hope that this model will be your guide.

FIGURE 1.4 A MODEL FOR MATHEMATIZING WORD PROBLEMS



The mathematizing sandbox involves three steps and two pauses:

Step 1 (Enter): Students' first step is one of reading comprehension. Students must understand the words and context involved in the problem before they can really dive into mathematical understanding of the situation, context, quantities, or relationships between quantities in the problem.

Pause 1: This is a crucial moment when rather than diving into an approach strategy, students make a conscious choice to look at the problem a different way, with a mind toward reasoning and sense making about the *mathematical story* told by the problem or context. You will notice that we often suggest putting the problem in your own words as a way of making sense. This stage is critical for moving away from the "plucking and plugging" of numbers with no attention to meaning that we so often see (SanGiovanni & Milou, 2018).

Step 2 (Explore): We call this phase of problem-solving stepping into the mathematizing sandbox. This is the space in which students engage their operation sense and play with some of the different representations mentioned earlier, making translations between them to truly understand what is going on in the problem situation. What story is being told? What are we comparing, or what action is happening? What information do we have, and what are we trying to find out? This step sometimes is reflected in mnemonics-based strategies such as STAR (stop, think, act, review) or KWS (What do you know? What do you want to know? Solve it.) or Pólya's (1945) four steps to problem solving (understand, devise a plan, carry out a plan, look back) or even CUBES. But it can't be rushed or treated superficially. Giving adequate space to the Explore phase is essential to the understanding part of any strategic approach. This is where the cognitive sweet spot can be found, and this step is what the bulk of this book is about.

Pause 2: The exploration done in the mathematizing sandbox leads students to the "a-ha moment" when they can match what they see happening in the problem to a known problem situation (Figures 1.2 and 1.3). Understanding the most appropriate problem situation informs which operation(s) to use, but it also does so much more. It builds a solid foundation of operation sense.

Step 3 (Express): Here students leave the sandbox and are ready to express the story either symbolically or in words or pictures, having found a solution they are prepared to discuss and justify.

Final Words Before You Dive In

We understand that your real life in a school and in your classroom puts innumerable demands on your time and energy as you work to address ambitious mathematics standards. Who has time to use manipulatives, draw pictures, and spend time writing about mathematics? Your students do! This is what meeting the new ambitious standards actually requires. It may feel like pressure to speed up and do more, but paradoxically, the way to build the knowledge and concepts that are currently described in the standards is by slowing down. Evidence gathered over the past 30 years indicates that an integrated and connected understanding of a wide variety of representations of mathematical ideas is one of the best tools in a student's toolbox (or sandbox!) for a deep and lasting understanding of mathematics (Leinwand et al., 2014). We hope that this book will be a valuable tool as you make or renew your commitment to teaching for greater understanding.