# Chapter I

## *What is Mathematical Investigation?*

Participants will need time to read the introductory paragraphs about the nature of mathematics and the rationale for investigation in the curriculum. Depending on the needs and interests of the group, they may feel a need to discuss these issues, but the section is quite full, and if the group has opportunities to talk outside the class, it may be best to get on with the work that these paragraphs introduce.

Here are some of the major ideas behind the module. Some may remain abstract and unclear until you have gone through the module. Though all should make sense after the module, not all are made explicit in the module.

- Being good at investigating is, in itself, a great asset in many and diverse fields: science, journalism, auto repair, medical diagnosis, history, law enforcement, and so on. It can be a useful tool in solving problems.
- Investigation in any arena—whether it is mathematics or any other field—involves looking deeply *into* the problem as it is presented, noticing which parts of the problem matter are central and which parts seem variable, extraneous, or of lesser importance. It requires paying attention also to what is assumed but *not* stated.
- Investigation also involves looking *beyond* the problem as it is presented. Truly thorough investigations involve what-ifs, what-if-nots, sub-problems, and side trips. A good investigator *looks for* problems and, in the pursuit of the problem at hand, may pose related new problems.
- One way to pose related new problems is to take the defining characteristics of the given problem and systematically vary them. By investigating what happens as the features of a problem are changed, one can get a good sense of the role played by each of those features. In science, this is simply called good experimenting—changing one feature in systematic ways while holding other features constant.

- Systematic experiments yield patterns. In mathematics, one *conjectures* that such patterns hold beyond the cases investigated, and tries to *prove* that they do. A complete mathematical investigation requires at least three steps: finding a pattern or other conjecture, seeking the logical interconnections that constitute proof, and organizing the results in a way that can be presented coherently.
- In the kinds of mathematics encountered before college, virtually *all* proofs provide much more than reassurance—the *way* they reassure is by showing *why* things are true, *how* they work.
- Because proofs involve assembling logically connected ideas in particular ways, sometimes the investigation of connected problems helps lead to proofs that might otherwise be elusive.
- Not all problems that can be posed are worthy of attention. Because one cannot follow every path that presents itself, one must make decisions about which problems to ignore and which to follow up; often one must make those decisions just on the basis of the features of the problem, before any investigative time is spent.

*Proof in mathematics beyond high school is not always so generous. Non-constructive proofs may provide assurance without yielding much insight into* why *things are as they are.*

This module does not try to extend a participant's knowledge in any particular *topic* in mathematics, like conic sections, or factoring trinomials, or trigonometric functions. It is about a way of analyzing problems, creatively posing problems, and proving conjectures.

The suggestions for the amount of time to spend on each problem give you a lot of freedom to explore (if you have a 2-hour class or workshop). We recommend taking 10–15 minutes or so for sharing among participants, especially following problems 1, 3, and 5. If there's still time, encourage participants to begin considering how they might justify the conjectures they've made. This is the goal of section 2, but it might prove useful to begin thinking about these issues ahead of time.

# 1. Problem solving and problem posing

## Problems presented in the text

**Problem 1 (10–15 minutes)**
**Goal:** To get enough of a preliminary sense of this problem to know how it might be modified by problem posing.

THE CONSECUTIVE SUM PROBLEM: *Which counting numbers can be expressed as the sum of two or more consecutive counting numbers?*

In the deliberately small amount of time allowed—just 10 to 15 minutes—nobody should expect to "solve" this problem, but adults typically come up with several initial conjectures. We just want to get them started thinking about the problem and its contexts. They'll be asked to analyze their conjectures in section 2. If you read ahead to section 2, you'll see that the "answer" is that counting numbers which are expressible as the sum of consecutive counting numbers are precisely those which are not powers of 2. Please do not "force" this conjecture out of the participants—it will occur to them soon enough. Some of the more common conjectures appear on the *Further Exploration* CD.

| *sum* | *series* |
|---|---|
| 2 | |
| 3 | $1+2$ |
| 4 | |
| 5 | $2+3$ |
| 6 | $1+2+3$ |
| 7 | $3+4$ |
| 8 | |
| 9 | $4+5$ |
| ⋮ | ⋮ |

***Table 1***

The task in problem 1 has a familiar enough structure to engage participants without much help or intervention from you. Problem 2 (below), by contrast, is generally quite unfamiliar, and participants may need help getting started.

| *series* | *sum* |
|---|---|
| $1+2$ | 3 |
| $2+3$ | 5 |
| $3+4$ | 7 |
| $4+5$ | 9 |
| ⋮ | ⋮ |
| $1+2+3$ | 6 |
| $2+3+4$ | 9 |
| $3+4+5$ | 12 |
| $4+5+6$ | 15 |
| ⋮ | ⋮ |

***Table 2***

To prepare yourself to help out with problem 2, notice that each of these tables represents a different way of looking at the problem—posing different sub-problems.

Tables 1 and 3 (in the margin) represent the least variance from the original problem. If you want to know which numbers you can make, list them (in order) and try (Table 1). The pattern, if it continues, already suggests that powers of 2 won't work.

Alternatively, one can list all possible sums (Table 3) and see what numbers they produce. Despite the systematic organization of this table, patterns don't stand out readily, but the fact that some numbers can be made in more than one way is sometimes more apparent from this table than with Table 1.

Table 2 is, in effect, an implicit posing of a new set of sub-problems: "What numbers can (can't) I express as sums of exactly *two* counting numbers?" Or *three*. Or *four*. It restricts a previously unrestricted feature of the problem ("two or more consecutive counting numbers").

| *high* | *series* | *sum* |
|---|---|---|
| 2 | $1+2$ | 3 |
| 3 | $1+2+3$ | 6 |
| | $2+3$ | 5 |
| 4 | $1+2+3+4$ | 10 |
| | $2+3+4$ | 9 |
| | $3+4$ | 7 |
| 5 | $1+2+3+4+5$ | 15 |
| | $2+3+4+5$ | 14 |
| | $3+4+5$ | 12 |
| | $4+5$ | 9 |
| 6 | ⋮ | ⋮ |

***Table 3***

***Problem:*** *List three more features of problem* 1.

**Problem 2 (10–15 minutes)**

**Goal:** To pose new problems, related to the original, that might shed light on the original or the landscape within which the original lies. Participants are not likely to have thought this way about problems. People often tend to ignore what seems "obvious," like the fact that the problem is about *sums*, which is why that one is given to them and why feature **e** ("no other restrictions") is given later. You may still have to help them not to ignore what seems too obvious to them.

Problem 1 has at least five essential features.

a. It is about a *sum*.
b. The sum contains *two or more* addends.
c. The addends must be *counting numbers*.
d. They must be *consecutive*.
e. There are restrictions that the problem *could* make, but *does not*. The fact that it *fails* to make more restrictions is part of what makes it *this* problem and not another.

*The familiar idea of "looking for special cases" is really a matter of noticing what restrictions the problem does not make, and adding them.*

Problem 3 asks participants to modify the original problem by altering one or more of these essential features. So that participants have a chance to explore before being given specific suggestions for changes, the *Ways to think about it* suggestions are placed later at the end of the corresponding section in the main text. But *you* need to be aware of these suggestions now in order to systematize the kinds of changes they might choose to make.

**Ways to think about it:**

*This is sometimes referred to as finding* special cases.

**i. Make a feature more restrictive:** If the problem is about triangles, restrict it to right (or scalene or . . . ) triangles. If the problem uses a calculation that involves two *or more* numbers, restrict it to *exactly* two (or three or . . . ).

*This is sometimes referred to as* generalizing, *or extending the domain.*

**ii. Relax a feature:** If the problem is about right triangles, see what happens to the problem when you allow *all* kinds of triangles, or maybe all polygons. If the problem specifies one set of numbers (e.g., {1, 2, 3, . . . }), see what happens when you allow all numbers to be used.

*These modifications may* change the domain *of a problem or alter a parameter.*

**iii. Alter the details of a feature:** If the problem is about right triangles, see what happens to the problem when you use some other kind of triangle. If the problem specifies one set of numbers (e.g., $\{1, 2, 3, \dots\}$), see what happens when you pose the problem with a different set of numbers, like $\{1, 3, 5, 7, \dots\}$ or $\{0, 3, 6, 9, 12, \dots\}$ or $\{0.5, 1, 1.5, 2, 2.5, \dots\}$. If the problem specifies particular arithmetic operations, see what happens if you systematically alter them (e.g., substituting $+$ and $-$ for $\times$ and $\div$ or vice versa), and if it specifies equality, see what happens if you require a specific inequality (e.g., $>$).

**iv. Check for uniqueness:** If a problem asks only *if* something can be done, ask if it can be done in only one way.

*Asking "How many ways can this be done?" is often productive.*

**Major goal:** A goal of the entire first two sections is for participants to see that pursuing the problem in depth—ferreting out its relatives and exploring them—is *much* richer than the original problem seems. Seeing the interconnectedness of mathematical results is essential to "understanding" mathematics.

**Homework:** *If your group does projects outside of workshop time, you might have them pick among the problems and investigate one or more of them in depth. Problem 3 gives some of the variants, and problem 4 outlines some of the territory to which these new variants lead.*

**Problem 3 (15 minutes)**

The following new problems all have interesting consequences related to the mathematics of grades 6–12. Solutions to these problems involve primes, odds, multiples of odds, factoring and the counting of certain factors, square numbers and the difference of squares $a^2 - b^2$, triangular numbers ($\frac{n(n+1)}{2}$) and the difference of triangular numbers, powers of 2, Pascal's Triangle, factorials, permutation numbers, and more.

***Problem:*** *Brainstorm to see what related problems evolve from this one as you change the features one (or at most two) at a time.*

The following discussion is identical to one that appears in the solutions on the *Further Exploration* CD, but we reproduce it here to assist the facilitation of this problem.

**i. Make a feature (like *b*) more restrictive:** Which counting numbers can be expressed as the sum of exactly 2 consecutive counting numbers? Exactly 3? Exactly $n$?

*Experiments like Table 2 are, in effect, playing with this restriction.*

**ii. Relax a feature:** For example, relax feature $c$, which specifies *counting* numbers as the domain. Instead, you could open the problem to *all* integers.

**iii. Alter the details of a feature:** The original problem requires consecutive counting numbers. What would happen if you chose consecutive *odd* numbers, such as $15 = 3 + 5 + 7$? Also, the problem does not restrict the starting point of each sum. What about requiring that the consecutive sums (or consecutive odd sums) *start with 1*? For example, look at the sums $1+2$, $1+2+3$, $1+2+3+4$, $\ldots$, or $1+3$, $1+3+5$, $1+3+5+7$, $\ldots$ . Altering details can even apply to the operation (feature $a$). What about consecutive products instead of consecutive sums? (Then re-consider all the previous variations!)

**iv. Check for uniqueness:** Are any numbers expressible as consecutive sums in more than one way? If so, is there a recognizable feature of those that can be expressed exactly *one* way? Two ways? Three ways? $n$ ways?

Each problem listed above offers something of real interest, but it is also possible to create problems that seem not to hold much promise, at least without adding enormous complexity. "Uninteresting" problems can be the result of changing too many features at once, or may result from making choices that seem foreign to the original context, or too arbitrary. Of course, some problems that *look* uninteresting may actually lead to deep results, but classroom time is not unlimited and one must often make decisions without first investigating deeply.

Here is a small list of kinds of problems that are occasionally suggested in workshops, but that seem less promising. Some are accompanied by reasons why they are poor bets.

*This could be answered as part of problem **i**. By itself, it is too arbitrary.*

**v. Restrict *b*:** Which counting numbers can be expressed as the sum of exactly 100 consecutive counting numbers?

*Too many changes. Change too arbitrary.*

**vi. Alter *b*, *c*:** Which numbers can be expressed as the sum of exactly 5 consecutive odd numbers?

*Primes are scattered too irregularly for "consecutive primes" to feel promising.* Adding*, in the context of primes, also makes the mathematics hard. In elementary mathematics, primes are multiplied, not added, to produce results.*

**vii. Change domain:** Instead of using all natural numbers, explore which numbers can be written as the sum of *consecutive prime* numbers (for example, $15 = 3+5+7$, or $10 = 2+3+5$).

**viii. Restrict the problem a different way:** Consider only consecutive sums (or consecutive even sums) starting with 8.

**ix. Alter the operation (change feature *a*):** Consider consecutive differences instead of consecutive sums, like $(((5-6)-7)-8)-9$.

*The result is mildly interesting, but the problem is so narrow that there's not much to find.*

**x. Alter features *a*, *b*, *c*:** Which numbers can be expressed as the difference of exactly two consecutive numbers from the set $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \ldots\}$?

Some problems are hard to judge in advance. Here are three.

**xi. Change domain:** Which *counting* numbers can be expressed as the sum of consecutive numbers from the set $\{\frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}, \frac{6}{3}, \dots\}$?

**xii. Change domain:** Which numbers can be expressed as sums of consecutive square numbers?

**xiii. Change domain:** Which numbers can be expressed as the sum of consecutive unit fractions $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots\}$?

**Problem 4 (10–15 minutes)**

We provide a detailed look at some possible conjectures in the *Further Exploration* CD, but do not include them here. Encourage participants to pick a "favorite" or "interesting" problem to work on (their choice).

***Problem:** Pick one or more of the problems you created in problem 3 and explore them just long enough to build some preliminary conjectures.*

**Problem 5 (15 minutes)**

Participants should now be more comfortable with the idea of describing, then altering, the features of a problem. They should start by listing the features (a la problem 2), then systematically alter any—or all—of them.

***Problem:** Apply these and your own rules to generate interesting variants on the following problem:*
*"How many triangles with perimeter 12 and integer side lengths can you construct?"*

**Problem 6 (15 minutes)**

This encourages participants to see that this process can be never-ending. The more we look at a problem or concept, the better we understand it and are able to "deconstruct" it further. This might also provide them with time to continue to consider how they might prove the general conjecture, since one approach is to consider separate, special, cases first.

***Problem:** Now, go back to the consecutive sums problem. Look over the list of features you made for problem 2 and see if applying these rules to each of the features gives you any new problems.*

## 2. You've got a conjecture—now what?

In the first section, participants investigated a rich problem about consecutive sums and explored ways to modify it and to pose new related problems. They also discussed why it is important to explore a problem beyond its solution. In this section, you'll continue this discussion. First, you'll investigate some important—and not so important—connections to which these modified problems lead. Second, you'll solve the modified problems, which will lead you to an explanation and proof of the main result (of the original consecutive sums problem).

### Problems presented in the text

***Problem:*** *Here are several pairs of variations on the original consecutive sums problem. Look at each pair, and try to decide, without first pursuing the problems, which choice seems more likely to lead somewhere.*

**Problems 1-5 (15–20 minutes)**
**Goal:** To develop participants' intuition—and some helpful rules of thumb—about which problems are more worthy of pursuit than others.

Get participants to discuss these questions within their groups. Remember that they're *not* being asked to prove these statements, they're being asked to determine which of each pair seems most likely to lead somewhere. Have the whole group share strategies and "rules of thumb." But don't let the time get away from you—you need to be sure there's enough time to get to problem 20.

***Problem:*** *How did you make your decision in each of the previous problems? What "rules of thumb" did you use to help distinguish between problems that are probably good and ones that are probably not worthwhile?*

**Problem 6 (10 minutes)**
This will be an excellent opportunity for group sharing and comparing. How do participants think about these problems? Is there agreement on each one, or is there an argument (of the intellectual variety)? This reflection activity, thinking about thinking, can be very helpful in focusing our efforts.

CONSECUTIVE SUM JEOPARDY: *In your investigation of the Consecutive Sums Problem, what questions (if any) have you run across which have the following sets of numbers as answers or partial answers?*

**Problem 7 (20–25 minutes)**
**Goal:** In the concrete terms of classroom topics, here mostly expressed as sets of numbers, participants get to see where the variations on the original problem lead.

Investigations of the sisters, cousins, and aunts of the CONSECUTIVE SUMS PROBLEM produce many sets of numbers as answers or partial answers. You may divide subproblems among the groups. Give each group a subset of 3-4 answers to determine for what problems they are answers. Have groups share their findings with the whole class at the end of this activity.

**Problems 8–20 (50–60 minutes)**

***Problems* 8–20:** *Find a proof or counterexample (an example that shows the statement is not always true) for each one of the following problems. These* 13 *problems form a path to a conclusion, so try to justify each statement.*

**Goals:** One goal is to encounter and practice explanatory proof in this non-geometric domain. Another is to see how these proofs build on each other to answer and explain the original consecutive sums problem.

It's hard to fight the urge to divide these problems among the groups, then have groups share their proofs with the rest of the class. However, the results *and* the proofs lead to the proof of problem 20, which *is* the main problem of this section, after all. Have teachers discuss how the preliminary problems assisted them in proving the original problem. If there's not enough time, take this up again during section 3—it's important! For hints on individual proofs, see the *Ways to think about it* section in the text or the solutions on the *Further Exploration* CD.

Possibly more important than any of these particular results is the idea that the sum of any number of consecutive counting numbers is the *product* of the number of terms being added and the *mean* of the first and last addends. Most of the participants will already be familiar with this idea, since it's the crux of Gauss's famous boyhood proof that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. In fact, this is true not only of the sum of consecutive counting numbers, but also the sum of consecutive terms in any arithmetic sequence. Participants will be asked to investigate this generalization in the *Further Exploration* materials.

*The story of Gauss's proof, while possibly apocryphal, is an integral part of mathematical lore. A version of the story appears in William Dunham's* Journey Through Tenius: The Great Theorems of Mathematics *(published by Penguin USA).*

If there's time, it would be very productive to take a few minutes to write out the complete proof of the CONSECUTIVE SUMS PROBLEM (they will have all the pieces, but they're spread out over the preliminary problems). If there's not time, take time at the beginning of section 3 or assign it as homework, if appropriate.

**Problem 21 (10 minutes)**

***Problem:*** *For each conjecture that you determined not to be true in problems* 8–19*, see if you can guess what* correct, *but incomplete, observations might have led to that conjecture.*

As in the other problems, the ideas of your participants will likely be varied. Encourage discussion and collegial critique rather than consensus.

For homework, participants could look at the variants of proof for a single formula for the sum of the first $n$ numbers. Section 4 starts with the discussion of these various proofs.

# 3. Do it yourself

During the first two sections, participants were involved in the investigation of a problem and also worked on problem posing and proof. This section asks participants to perform investigations of two more problems on their own, repeating the procedures of the previous sections: analyzing and altering the original problem in order to pose further problems and gain a deeper understanding of its context.

### Problems presented in the text

THE POST OFFICE PROBLEM *A particularly quirky post office clerk sells only* 7*-cent stamps and* 9*-cent stamps. Can exactly* 32*-cents' worth of postage be made using these stamps? Can 33 cents be made? Which amounts, if any, cannot be made?*

THE PYTHAGOREAN THEOREM: *This theorem, central to an enormous amount of mathematics, can be thought of as a statement about shapes. "The (area of the) square on the hypotenuse of a right triangle is equal to the sum of the (areas of the) squares on the two legs of that triangle." Alternatively, it can be thought of algebraically, as a much more generic statement about the way some numbers are related. "The sum of two squares is equal to another square," is often written* $a^2 + b^2 = c^2$.

**Goal:** Participants practice problem-posing techniques, applying them to new problem situations. They should realize that these are *general* techniques, useful for many problem situations, and that problem *posing* is a piece of mathematical creativity that is often also helpful in problem *solving*.

These two situations have been selected for practice because they are, themselves, quite rich and because their variants can therefore also be expected to be of value. The Pythagorean Theorem is, of course, ubiquitous. The post office problem is beginning to appear, in various forms, in many curricula. You could give these two problems to different groups of teachers, because they might not be able to finish both of the problems during the session. Try to leave time for presentations and discussion.

So, should you require all participants to work on both problems, or should you ask them to pick one and spend the session working on it? Or should you ask them to spend 10 minutes on each problem and decide which one they want to pursue? It's up to you, depending on the experiences and needs of your participants and the particular circumstances in which you meet them. In fact, you may want to stretch this session into two by including work on the justification of the solution one—or both—of these problems and/or their cousins.

No matter how you decide to work on these materials, it is *essential* that participants be given ample opportunity to talk with each other while working on the problem *and* share their work with you and the rest of the group.

Be sure to refer to the *Further Exploration* materials for possible directions participants might take while analyzing and altering these problems.

## 4. You know the answer? Prove it.

Previous sections asked participants to consider a variety of problem-solving and problem-posing situations, with some proofs thrown in for good measure. This section continues with the important theme of mathematical justification, requiring teachers to carefully read, correct (if necessary), and then alter a set of different proofs of the same fact:

For all counting numbers $n$, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

### Problems presented in the text

**Problem 1 (20–30 minutes for each proof)**
You might find it useful to divide up the proofs among groups of teachers and then have them make presentations to the whole class or workshop. The specific make-up of the teachers (preservice or inservice, level of experience with proof or advanced mathematics) plays a big role in how to proceed. If they are not familiar with formal proof or haven't had to think about such issues in a long time, you might decide to focus on fewer parts (among *i*–*iv*), concentrating on depth rather than breadth. Alternatively, you could forego part (b) and/or part (c) on individual proofs.

Sometimes, there is a lot of rewriting necessary when proving alternate conjectures (parts (b) and (c)), and other times, there's not. On some occasions, the given proof method is not obviously adaptable (for example, you might be hard pressed to use the method of proof *i*, considering odd and even cases, to prove a formula for the sum of odd counting numbers).

***Problem:*** *As you read each alleged proof, do the following:*

*(a) Decide whether the argument is a genuine, acceptable proof. If you feel it is not, fix it.*

*(b) Rewrite the argument to make it fit a conjecture about the sums of consecutive* odd *numbers starting at* 1.

*(c) What if the numbers were not consecutive counting numbers but, say, consecutive multiples of* 3*, or not starting at* 1*, or ...*

**Problem 2 (remaining time or later class)**
The particular format of the presentations is, of course, up to you. If possible, give participants additional time to work together on their problems and the planning of their presentations, then have them present their results to the class at a future meeting.

***Problem:*** *You have generated many problems and partial or complete results to some of them (in this session, as well as earlier ones). Pick one, or a group of closely related ones, and organize a presentation of your work.*

## 5. Discerning what *is*, predicting what *might be*

This section might, at first glance, seem unrelated to the previous sections, but statistical investigation is an important aspect of mathematical problem solving. As it is a very different "habit of mind" from typical problem solving, it is worth spending time on its development. As participants learn in this section, the method chosen to display data is a significant one. While it may be true "the data don't lie," it's certainly even more true that the choice of data presentation can affect the interpretation by the data "consumer." Therefore, all teachers and students must be aware of the effect presentation plays on our interpretation of data.

### Problems presented in the text

***Problem:*** *Work through problem* **A***. Also, decide*
*i. which items seem to require little more than using definitions and procedures;*
*ii. which items require some judgment as well;*
*iii. which items are ambiguous or meaningless.*

**Problem 1 (15 minutes)**
Problem **A** is copied below for your convenience:

**A.** To help decide what kinds of items to keep in stock, a store kept track of the ages of its customers. This stem-and-leaf plot shows the data for one 15-minute period.

| | |
|---|---|
| 3 | 3 4 8 9 |
| 2 | 5 8 8 8 8 8 9 9 9 |
| 1 | 0 1 1 1 2 2 3 4 4 |
| 0 | 6 7 8 |

**(1)** How many people entered the store during that 15 minutes?
**(2)** Which is the most common age group?
**(3)** Five customers were the same age. How old were they?
**(4)** Is 25 a typical age for a customer?

Don't have teachers spend more than about 5 minutes on these, unless participants are unfamiliar with stem-and-leaf plots. Use the bulk of the time to share and compare solutions. Some participants might not see the ambiguity inherent in certain questions, thinking, "I know what they mean." It's worth taking time to consider these preconceptions. After all, while everyone might "know what they mean," that meaning may very well be different from one individual to the next.

***Problem:*** *Plot the data as histograms in the two specified ways.*

**Problem 2 (5 minutes)**
This shouldn't take very long; the discussion concerning interpretation of what is seen in the two tables is part of problem 3. You might even want to complete these tables as a whole class by having a participant fill out the tables on an overhead. Be

sure that participants are careful—you don't want the misrepresentation of data to result from haphazard data recording.

**Problem 3 (10 minutes)**
Encourage teachers to realize these tables illustrate the fact that the choices made in displaying the data can greatly affect the interpretation of the data. The statement, "The data speak for themselves" is very misleading, not to mention untrue!

As part of the analysis for this problem, ask teachers to check to see whether any of their solutions to problem 1 would change with the second table. Have them focus on the most common age group if they have trouble getting anywhere.

***Problem:*** *What patterns of customer ages are "revealed" in these two histograms, and what inference might you draw about the clientele of the store based on each pattern? Which corresponds to the stem-and-leaf plot? What, if anything, about the data might you use to help decide which pattern better reflects the truth about the store's customers?*

**Problem 4 (15 minutes or less)**
Problem **B** is copied below for your convenience:

**B.** Students recorded their weights to the nearest pound as they tried out for the school's track team. This is the full list: 138, 103, 135, 115, 143, 105, 112, 115, 125, 150, 125, 120, 101, 152, 149, 152, 137, 114, 119, 128, 125, 104, 110, 108, 144, 115, 144, 125, 133, 136, 144, 117, 125, 132.
**(1)** Make a stem-and-leaf plot to display these weights.
**(2)** Find the mean, median, mode, and range of these data.
**(3)** Describe a pattern you see in the data.

***Problem:*** *The box contains another problematic problem. Again, as you work through it, decide*
*i. which items seem to require nothing more than definitions and procedures;*
*ii. which items require some judgment as well;*
*iii. which items are ambiguous or meaningless.*

Quick "accuracy check": be sure that everyone has 34 data points.

**Problem 5 (5–10 minutes)**
Be sure that participants realize the histograms are horizontal, rather than vertical.

***Problem:*** *After working through all parts of problems **B** and 4, make histograms in the two ways called for.*

**Problems 6-7 (15 minutes)**
Most observations will mirror those in problem 3, but take time to see whether anything new comes up. Be sure that at least part of the discussion focuses on the fact that the display choices affect the message received.

***Problem:*** *What do the histograms "reveal" about the data?*
***Problem:*** *What do the histograms reveal about* histograms*?*

**Problems 8–10 (15–20 minutes)**
In problem 8, teachers will recognize that the varied parameter in previous histograms was the "location" of each interval (or bin)—this might be expressed as the center of the interval or as either the minimum (left endpoint) or the maximum (right endpoint) value in the interval.

***Problem:*** *Problem* 5 *asked you to make two histograms. A change in one parameter distinguished the two histograms. What was varied?*

***Problem:*** *Experiment with interval width in your head. When you change this parameter, how does it affect the visual pattern you see?*

***Problem:*** *Is there a way to determine the "right" interval width for a particular data set?*

Have participants imagine "extreme" cases in problem 9. For instance, the narrowest possible interval that would make sense would have width 1, while the widest width could be 52 (encompassing everything from 101 to 152).

In problem 10, the "right" interval width is unknowable, in general. However, a case could be made for interval width 1, since then there is no "clumping," and therefore, it might give rise to the least amount of misconception. Determining the "right" interval width for a particular data set is not a mathematical process: There is no mathematical reason to support any one choice over any other. The "right" choice is in the eye of the beholder—an interpretive choice that is subject (like much of statistical argumentation) to debate by others who hold a different point of view.

***Problem:*** *Here are three ways a student might think about the request to "find the mode" in problem* **B2**. *Each answer is based on a different interpretation of "find the mode." What interpretation leads to each answer? What is* correct *about each interpretation?*
*(a) The mode is* 125.
*(b) The mode is* 110 *to* 119.
*(c) People's actual weights can't have a mode.*

**Problem 11 (5–10 minutes)**

Encourage participants to come up with valid, or at least reasonable, rationales for each response—perhaps it will help to repeat the maxim, "Incorrect answers are often the correct answer to a different question." Have them look for what is *correct* about each answer.

Of course, part (a) gives the standard response, since 125 is the most common weight listed in the data. Part (b) gives the most represented decade, as exhibited in the stem-and-leaf plot, but it might be reasonable to think of clumping some data together, in general. Part (c) is probably most difficult to justify, but it's possible the student could be thinking about the fact that it's very unlikely that two people have *exactly* the same weight.

**Problems 12–21 (as time allows)**

It might not be possible for all participants to finish all of the remaining problems within a two-hour time period. Perhaps the problems could be divided up between groups of teachers, who will present their solutions and lead discussions during a subsequent class.

***Problem:*** *Suppose that the weights had been recorded to the nearest tenth of a pound, instead of to the nearest pound. Further suppose that the mean, median, mode, and range were calculated with these new, more accurate data. By how much,* at the very most*, could each of the measures differ from the ones computed with the nearest-pound data?*

**Problem 12**

This might be a good problem for a whole-group discussion. Start with small groups, and have them share after discussing the problem for a while.

Any of the weights could be rounded up or down by as much as half a pound. For example, if someone's actual weight, $w$, satisfied $101.5 \leq w < 102.5$, it would have been recorded as 102 in the original list. There are some potential misconceptions here. For instance, if someone's weight is 101.45, their weight to the nearest tenth of a pound would be 101.5, but their weight to the nearest pound is 101—not 102.

Therefore, the mean, median, and range could go up or down by as much as a half pound (so the possible values lie within a pound of each other), while the mode could change dramatically, going as high as 149.5 (if the weights originally recorded as 149 and 150 could both be rounded to 149.5) and as low as 114.5 (if the weights at 114 and 115 all rounded to 114.5).

**Problems 13–21**

See the hints in the *Ways to think about it* section of the text or the solutions on the *Further Exploration* CD.